Mostow's Decomposition Theorem for L^*-groups and Applications to affine coadjoint orbits and stable manifolds

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Abstract

Mostow's Decomposition Theorem is a refinement of the polar decomposition. It states the following. Let G be a compact connected semi-simple Lie group with Lie algebra $\mathfrak{g}$. Given a subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that $[X, [X, Y]] \in \mathfrak{h}$ for all X, Y in $\mathfrak{h}$, the complexified group $G^{\mathbb{C}}$ is homeomorphic to the product $G \cdot \exp i\mathfrak{m} \cdot \exp i\mathfrak{h}$, where $\mathfrak{m}$ is the orthogonal of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Killing form. This Theorem is related to geometric properties of the non-positively curved space of positive-definite symmetric matrices and to a characterization of its geodesic subspaces. The original proof of this Theorem given by Mostow in [14] uses the compactness of G. We give a proof of this Theorem using the completeness of the Lie algebra $\mathfrak{g}$ instead, which can therefore be applied to an L^*-group of arbitrary dimension. A different proof of this generalization of Mostow's Decomposition Theorem has been obtained independently by G. Larotonda in [13]. Some applications of this Theorem to the geometry of (affine) coadjoint orbits and stable manifolds are given.

Résumé

Le théorème de décomposition de Mostow est un raffinement de la décomposition polaire. Il s'énonce comme suit. Soit G un groupe de Lie compact connexe semi-simple d'algèbre de Lie $\mathfrak{g}$. Étant donné un sous-espace vectoriel $\mathfrak{h}$ de $\mathfrak{g}$ tel que $[X, [X, Y]] \in \mathfrak{h}$ pour tout X, Y dans $\mathfrak{h}$, le groupe de Lie complexifié $G^{\mathbb{C}}$ de G est homéomorphe au produit $G \cdot \exp i\mathfrak{m} \cdot \exp i\mathfrak{h}$, où $\mathfrak{m}$ est l'orthogonal de $\mathfrak{h}$ dans $\mathfrak{g}$ relativement à la forme de Killing. Ce théorème repose sur la géométrie à courbure négative de l'espace des matrices symétriques définies positives, et sur la caractérisation de ses sous-espaces totalement géodésiques. La preuve initiale de ce théorème donnée par Mostow dans [14] utilise la compacité de G. Nous en donnons une démonstration qui utilise seulement la complétude de l'algèbre de Lie $\mathfrak{g}$, et qui s'applique de ce fait à un L^*-groupe de dimension arbitraire. Une autre démonstration de cette généralisation du théorème de décomposition de Mostow a été obtenue indépendamment par G. Larotonda dans [13]. Quelques applications de ce théorème à la géométrie des orbites coadjointes (affines) et des surfaces stables sont données.

Contents

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1 Mostow's Decomposition Theorem for L^*-groups

1.1 Introduction

This section is devoted to a proof of Mostow's Decomposition Theorem for a separable L^*-group of finite or infinite dimension. Some general results in the geometry of Banach manifolds are stated passing by, but have their interest on their own (for instance Proposition 1.9). We refer to [14] for the original arguments in the finite-dimensional case, to [13] for a different proof in the infinite-dimensional setting and to [1] for a generalization to some von Neumann algebras. Let us first state the Theorem :

Theorem 1.1 *Let G be a semi-simple connected L^*-group of compact type with Lie algebra $\mathfrak{g}$, $G^{\mathbb{C}}$ the connected L^*-group with Lie algebra $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \oplus i\mathfrak{g}$, E a closed subspace of $i\mathfrak{g}$ such that*

$$[X, [X, Y]] \in E, \qquad \textit{for all} \quad X, Y \in E,$$

and F the orthogonal of E in $i\mathfrak{g}$. Then $G^{\mathbb{C}}$ is homeomorphic to the product $G \cdot \exp F \cdot \exp E$.

This section is organized as follows. First we investigate the geometry of the space $\mathcal{P}$ of positive-definite self-adjoint operators in the group $\mathrm{GL}_2(\mathcal{H})$ of invertible operators which differ from the identity by Hilbert-Schmidt operators. We show in particular that it is a symmetric space of non-positive sectional curvature and that the exponential map defined by the usual power series is a diffeomorphism from the space $\mathcal{S}_2(\mathcal{H})$ of Hilbert-Schmidt self-adjoint operators onto $\mathcal{P}$. Moreover we show that the exponential map is the Riemannian exponential map at the identity with respect to the defined metric on $\mathcal{P}$. This is implied by a general result on the geodesics in locally symmetric spaces that we state in the more general context of Banach manifolds (Proposition 1.9). This study implies the usual Al-Kashi inequality on the sides of a geodesic triangle in the non-positively curved space $\mathcal{P}$, and the convexity property of the distance between two geodesics. In the second subsection, a characterization of the geodesic subspaces of $\mathcal{P}$ is given which mainely follows [14]. In subsection 1.4, the key-step for the proof of Mostow's Decomposition Theorem is given by the construction of a projection from $\mathcal{P}$ to every closed geodesic subspace. The arguments given here for the existence of such projection are simpler and more direct then the ones given in the original paper [14], and apply to arbitrary dimension. In the last subsection, we use this projection to prove the Theorem stated above. For examples of application of this Theorem we refer to sections 2 and 3 of the present work.

1.2 The space $\mathcal{P}$ of positive-definite self-adjoint operators in $\mathrm{GL}_2(\mathcal{H})$

Let $\mathcal{H}$ be an Hilbert space of arbitrary dimension. The group $\mathrm{GL}_2(\mathcal{H})$ is the group of invertible operators which differs from the identity by an Hilbert-Schmidt operator :

$$\mathrm{GL}_2(\mathcal{H}) := \{g \in \mathrm{GL}(\mathcal{H}) \mid a - \mathrm{id} \in L^2(\mathcal{H})\},$$

where $L^2(\mathcal{H})$ denotes the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}$, i.e. the set of operators A on $\mathcal{H}$ such that the trace of A^*A is finite, endowed with the Hermitian scalar product :

$$\langle A, B \rangle := \mathrm{Tr}\, A^* B.$$

The Lie-algebra of $\mathrm{GL}_2(\mathcal{H})$ is the Hilbert space $L^2(\mathcal{H})$ endowed with the commutator of operators, and will be denoted by $\mathfrak{gl}_2(\mathcal{H})$. It is an L^*-algebra for the involution $*$ which maps an operator to its adjoint, in the sense that the bracket on $\mathfrak{gl}_2(\mathcal{H})$ and the involution $*$ are related by the following property :

$$\langle [x\,,\, y]\,,\, z \rangle = \langle y\,,\, [x^*,\, z] \rangle$$

for every x, y and z in $\mathfrak{gl}_2(\mathcal{H})$. We define also the unitary group $\mathrm{U}_2(\mathcal{H})$ and its Lie algebra $\mathfrak{u}_2(\mathcal{H})$:

$$\begin{aligned} &\mathrm{U}_2 = \{a \in \mathrm{U}(\mathcal{H}) \mid a - \mathrm{id} \in L^2(\mathcal{H})\}, \\ &\mathfrak{u}_2 = \mathfrak{u}(\mathcal{H}) \cap L^2(\mathcal{H}). \end{aligned}$$

The Hilbert space $\mathfrak{gl}_2(\mathcal{H})$ splits into the direct sum of $\mathfrak{u}_2(\mathcal{H})$ and the closed linear subspace $\mathcal{S}_2(\mathcal{H})$ of self-adjoint elements in $\mathfrak{gl}_2(\mathcal{H})$. The exponential map defined as

$$\exp(A) := \sum_{n=0}^{+\infty} \frac{A^n}{n!} \tag{1}$$

for all A in $\mathfrak{gl}_2(\mathcal{H})$, takes $\mathcal{S}_2(\mathcal{H})$ to the submanifold $\mathcal{P}$ of $\mathrm{GL}_2(\mathcal{H})$ consisting of positive-definite self-adjoint operators :

$$\exp\ : \mathcal{S}_2(\mathcal{H}) = \{A \in \mathfrak{gl}_2(\mathcal{H}), A^* = A\} \longrightarrow \mathcal{P} = \{A \in \mathcal{S}_2(\mathcal{H}), A^*A > 0\}.$$

Note that for $\mathfrak{a} \in \mathfrak{gl}_2(\mathcal{H})$, the curve $\gamma(t) := \exp(t\mathfrak{a})$ satisfies $\dot\gamma(t) = \left(L_{\gamma(t)}\right)_*(\mathfrak{a})$, where $L_{\gamma(t)}$ denotes left translation by $\gamma(t)$, hence the exponential map defined by (1) is the usual exponential map on the Lie group $\mathrm{GL}_2(\mathcal{H})$. Note also that the tangent space to $\mathcal{P}$ at p is obtained from $\mathcal{S}_2(\mathcal{H})$ by left or right translation by p. Let us endowed $\mathcal{P}$ with the following Riemannian metric :

$$\mathrm{g}_p(U,V) = \mathrm{Tr}\,\left(p^{-1}Up^{-1}V\right).$$

Note that for $U, V \in T_p\mathcal{P}$, $p^{-1}U$ and $p^{-1}V$ belong to $\mathcal{S}_2(\mathcal{H})$, thus the product $p^{-1}Up^{-1}V$ is of trace class, so g is well-defined and positive-definite. The identity on $\mathcal{H}$ belongs to $\mathcal{P}$ and as such will be denoted by o in order to avoid confusions with identity operators on other Hilbert spaces.

Proposition 1.2 *The action of $GL_2(\mathcal{H})$ on $\mathcal{P}$ defined by :*

$$\begin{array}{rcl} GL_2(\mathcal{H}) \times \mathcal{P} & \rightarrow & \mathcal{P} \\ (\,x\,,\,p\,) & \mapsto & x\cdot p = x^*p\;x, \end{array}$$

is a transitive action by isometries.

□ **Proof of Proposition** 1.2:
For every p in $\mathcal{P}$, the square root of p is well-defined and belongs to $\mathcal{P}$. In other words there exists q in $\mathcal{P}$ such that $p = q^2$. But $q^* = q$, hence $p = q^*q = q\cdot o$, and the transitivity follows. To show that $\mathrm{GL}_2(\mathcal{H})$ acts by isometries, for x in $\mathrm{GL}_2(\mathcal{H})$ and $p \in \mathcal{P}$, let us denote by x_* the differential of x at p. By linearity, one has $x_*(U) = x^*Ux$ for every $U \in T_p\mathcal{P}$. It follows that

$$\begin{aligned} \mathrm{g}_{x\cdot p}\left(x_*(U)\,,\,x_*(V)\right) &= \mathrm{Tr}\,(x\cdot p)^{-1}x_*(U)(x\cdot p)^{-1}x_*(V) = \mathrm{Tr}\,x^{-1}p^{-1}(x^*)^{-1}x^*Uxx^{-1}p^{-1}(x^*)^{-1}x^*Vx \\ &= \mathrm{Tr}\,x^{-1}p^{-1}Up^{-1}Vx = \mathrm{Tr}\,p^{-1}Up^{-1}V = \mathrm{g}_p(U,V), \end{aligned}$$

and the action of $\mathrm{GL}_2(\mathcal{H})$ on $\mathcal{P}$ is an action by isometries. □

Remark 1.3 Since the stabilizer of the identity is the unitary group $\mathrm{U}_2(\mathcal{H})$, it follows that $\mathcal{P} = \mathrm{GL}_2(\mathcal{H})/U_2(\mathcal{H})$ as homogeneous Riemannian manifold.

Let us recall the following definition.

Definition 1.4 A Riemannian Banach manifold $\mathcal{P}$ is called *symmetric* if for every p in $\mathcal{P}$, there exists a globally defined isometry s_p which fixes p and such that the differential of s_p at p is $-\mathrm{id}$.

Proposition 1.5 *The manifold $\mathcal{P}$ is a symmetric homogeneous Riemannian manifold of non-positive sectional curvature.*

□ **Proof of Proposition** 1.5:
Consider the inversion $i\,:\,\mathcal{P} \rightarrow \mathcal{P}$ defined by $i(p) = p^{-1}$. An easy computation shows that the differential i_* of i at $p \in \mathcal{P}$ takes each vector $U \in T_p\mathcal{P}$ to $i_*(U) = -p^{-1}Up^{-1}$. One therefore has :

$$\begin{aligned} \mathrm{g}_{i(p)}\left(i_*(U), i_*(V)\right) &= \mathrm{g}_{p^{-1}}\left(-p^{-1}Up^{-1}, -p^{-1}Vp^{-1}\right) = \mathrm{Tr}\,p(-p^{-1}Up^{-1})p(-p^{-1}Vp^{-1}) \\ &= \mathrm{Tr}\,Up^{-1}Vp^{-1} = \mathrm{Tr}\,p^{-1}Up^{-1}V = \mathrm{g}_p(U,V), \end{aligned}$$

hence i is an isometry of $\mathcal{P}$. Since i fixes the element $e = \mathrm{id}_{\mathcal{H}}$ and since the differential of i at e is $-\mathrm{id}$ on $T_e\mathcal{P} = \mathcal{S}_2(\mathcal{H})$, it follows that i is a global symmetry with respect to e. It follows from the transitive action of $\mathrm{GL}_2(\mathcal{H})$ that for every p in $\mathcal{P}$ there exists a global isometry s_p which fixes p and whose differential at p is $-\mathrm{id}$ on $T_p\mathcal{P}$, namely $s_p(x) = p\,x^{-1}\,p$. Whence $\mathcal{P}$ is a symmetric homogeneous Riemannian manifold.

Since the $\mathrm{GL}_2(\mathcal{H})$-invariant metric on $\mathcal{P}$ is a *strong* Riemannian metric (this means that g induces an isomorphism between the tangent space at $p \in \mathcal{P}$ and its continuous dual, which is clearly the case at e hence everywhere), the Levi-Civita connection is well-defined by Koszul formula (see for instance Theorem 3.1 page 54 in [2] where this formula is recalled). The computation of the curvature tensor R is therefore step by step the same as in the finite-dimensional case and provides that the sectional curvature K_o at o is given by :

$$K_o(X,Y) := \frac{\mathrm{g}\left(R_{X,Y}X, Y\right)}{\mathrm{g}(X,X)\mathrm{g}(Y,Y) - \mathrm{g}(X,Y)^2} = \frac{\mathrm{g}\left([[X,Y],X]\,,Y\right)}{\mathrm{g}(X,X)\mathrm{g}(Y,Y) - \mathrm{g}(X,Y)^2},$$

for all X, Y in $T_o\mathcal{P} = \mathcal{S}_2(\mathcal{H})$ (see Proposition 7.72 page 193 in [4], or Proposition 6.5 page 92 in [2]). Now the sign of the sectional curvature of the 2-plane generated by X and Y is the sign of $\mathrm{g}\left([[X,Y],X],Y\right)$. By definition of g, one has :

$$\begin{aligned} \mathrm{g}\left([[X,Y],X],Y\right) &= \mathrm{Tr}\;[[X,Y],X]\,Y = \mathrm{Tr}\;([X,Y]XY - X[X,Y]Y) \\ &= \mathrm{Tr}\,[X,Y][X,Y] = -\mathrm{Tr}\,[X,Y]^*[X,Y] \leq 0, \end{aligned}$$

the last identity following from the fact that $[X,Y]$ belongs to $[\mathcal{S}_2(\mathcal{H}), \mathcal{S}_2(\mathcal{H})] \subset \mathfrak{u}_2(\mathcal{H})$. □

The following Proposition is well-known in the theory of linear group. The reader will find the computation of the differential of the exponential using powers series as a consequence of Lemma 1 in [14] (this computation works as well in the infinite-dimensional setting, see for instance Proposition 2.5.3 page 116 in [20]). See also [9].

Proposition 1.6 *For every Hilbert Lie group G, with Lie algebra $\mathfrak{g}$, the differential of the exponential map* $\exp\ : \mathfrak{g} \rightarrow G$ *is given at $X \in \mathfrak{g}$ by :*

$$(d_X \exp)\,(Y) = L_{\exp(X)}\left(\frac{1 - e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}\right)(Y). \tag{2}$$

for all Y in $\mathfrak{g}$.

□ **Proof of Proposition** 1.6:
Let us define the following map :

$$\begin{array}{cccc} \Phi\ : & \mathbb{R}^2 & \longrightarrow & G \\ & (t,s) & \longmapsto & \exp\left(t(X+sY)\right). \end{array}$$

Consider the push-forward U and V of the vector fields $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ on $\mathbb{R}^2$:

$$U\left(\Phi(t,s)\right) := \Phi_*\left(\frac{\partial}{\partial t}\right) \quad \text{and} \quad V\left(\Phi(t,s)\right) = \Phi_*\left(\frac{\partial}{\partial s}\right).$$

Denote by $[\cdot\,,\,\cdot]_{\mathfrak{X}}$ the bracket of vector fields. One has:

$$[U,V]_{\mathfrak{X}} = \left[\Phi_*\left(\frac{\partial}{\partial t}\right), \Phi_*\left(\frac{\partial}{\partial s}\right)\right]_{\mathfrak{X}} = \Phi_*\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right]_{\mathfrak{X}} = 0. \tag{3}$$

Note that

$$V\left(\Phi(t,s)\right) = \frac{\partial \Phi}{\partial s}(t,s) = \left(d\exp_{(tX+stY)}\right)(tb) \quad \text{and} \quad V\left(\Phi(1,0)\right) = (d_X \exp)\,(Y).$$

The idea of this proof is to explicit the differential equation satisfied by the $\mathfrak{g}$-valued function

$$v(t) := \left(L^{-1}_{\Phi(t,0)}\right)_* V\left(\Phi(t,0)\right) = \left(L^{-1}_{\exp(tX)}\right)_* V\left(\exp(tX)\right).$$

For this purpose let us introduce the connection ∇ on the tangent bundle of G for which the left-invariant vector fields are parallel. It is a flat connection since a trivialization of the tangent bundle is given by the left-invariant vector fields associated to an Hilbert basis of $\mathfrak{g}$. Note that by the very definition of the exponential map on a Lie group, $t \mapsto \exp(tX) = \Phi(t,0)$ is a geodesic for this connection, hence

$$\nabla_U U = 0 \tag{4}$$

along $\Phi(t,0)$. More generally, the connection ∇ can be expressed using the (left) Maurer-Cartan $\mathfrak{g}$-valued 1-form defined by

$$\theta_g(Z) = (L_g)^{-1}_*(Z).$$

Indeed, for a vector field W and a tangent vector Z in T_gG, one has

$$\left(\nabla_Z W\right)(g) := (L_g)_*\left(Z \cdot \theta(W)\right), \tag{5}$$

where $Z \cdot \theta(W)$ denotes the derivative along the vector Z of the $\mathfrak{g}$-valued function $\theta(W)$. Let us denote by T and R the torsion and the curvature of ∇. By definition :

$$T(U,V) := \nabla_U V - \nabla_V U - [U,V]_{\mathfrak{X}}$$

$$\text{and} \qquad R_{U,V}U := \nabla_V \nabla_U U - \nabla_U \nabla_V U - \nabla_{[V,U]_{\mathfrak{X}}}.$$

By (3), one has

$$\nabla_U V = \nabla_V U + T(U,V),$$

hence

$$\nabla_U \left(\nabla_U V\right) = \nabla_U \nabla_V U + \nabla_U T(U,V).$$

But the curvature tensor vanishes, hence (3) and (4) imply

$$\nabla_U \nabla_V U = \nabla_V \nabla_U U - \nabla_{[V,U]_{\mathfrak{X}}} = 0.$$

Consequently one has

$$\nabla_U \left(\nabla_U V\right) = \nabla_U T(U,V).$$

By the expression (5) of the connection, one has

$$T(U,V)\left(\Phi(t,0)\right) = \left(\nabla_U V - \nabla_V U\right)\left(\Phi(t,0)\right) = \left(L_{\Phi(t,0)}\right)_* \left(U \cdot \theta(V) - V \cdot \theta(U)\right).$$

Let us recall that the torsion is a tensor, hence $T(U,V)\left(\Phi(t,0)\right)$ does not depend on the extensions of the vectors $U\left(\Phi(t,0)\right)$ and $V\left(\Phi(t,0)\right)$ into vector fields. Using the left-invariant extensions of these two vectors one see easily that by the very definition of the bracket in the Lie algebra $\mathfrak{g}$ one has

$$T(U,V)(\Phi(t,0)) = -\left(L_{\Phi(t,0)}\right)_* [\theta(U), \theta(V)].$$

Whence

$$\nabla_U \left(\nabla_U V\right) = -\nabla_U \left(L_{\Phi(t,0)}\right)_* [\theta(U), \theta(V)] = \left(L_{\Phi(t,0)}\right)_* \frac{d}{dt}[\theta(U), \theta(V)].$$

Now, along $\Phi(t,0)$, the vector $\theta(U)$ is the constant vector X, and $\theta(V) = v(t)$. It follows that

$$\frac{d^2 v(t)}{dt^2} = \left(L^{-1}_{\Phi(t,0)}\right)_* \nabla_U \left(\nabla_U V\right) = -\left(L^{-1}_{\Phi(t,0)}\right)_* \nabla_U \left(L_{\Phi(t,0)}\right)_* [X, \theta(V)] = -\nabla_U [X, \theta(V)].$$

This leads to the following differential equation

$$\frac{d^2 v(t)}{dt^2} = -\left[X, \frac{dv}{dt}\right]$$

with initial conditions $v(0) = 0$ and $\frac{dv}{dt}_{|t=0} = Y$. A first integration leads to

$$\frac{dv}{dt} = e^{-t\,\mathrm{ad}(X)}(Y)$$

and a second to

$$v(t) = \left(\frac{1 - e^{-t\,\mathrm{ad}(X)}}{\mathrm{ad}(X)}\right)(Y).$$

So the result follows from the identity $v(1) = \left(L^{-1}_{\exp(X)}\right)_* \left(d_X \exp\right)(Y)$. □

Corollary 1.7 *The exponential map is a diffeomorphism from $\mathcal{S}_2(\mathcal{H})$ onto $\mathcal{P}$.*

□ **Proof of Corollary** 1.7:
For X in $\mathcal{S}_2(\mathcal{H})$, let us define the following map :

$$\begin{array}{rccl} \tau_X : & \mathcal{S}_2(\mathcal{H}) & \longrightarrow & \mathcal{S}_2(\mathcal{H}) \\ & Y & \mapsto & \tau_X(Y) := L_{\exp(-\frac{X}{2})} R_{\exp(-\frac{X}{2})} d_X \exp(Y). \end{array}$$

Using the notation $D_X := \mathrm{ad}(X)$ and

$$\frac{\exp\left(\frac{D_X}{2}\right) - \exp\left(-\frac{D_X}{2}\right)}{D_X} = \frac{\sinh\left(D_X/2\right)}{D_X/2} = \sum_{n=0}^{+\infty} \frac{(D_X/2)^{2n}}{(2n+1)!},$$

we have as a direct consequence of formula (2) in Proposition 1.6 that for all Y in $\mathcal{S}_2(\mathcal{H})$

$$\tau_X(Y) = \frac{\sinh(D_X/2)}{D_X/2}(Y),$$

Every X in $\mathcal{S}_2(\mathcal{H})$ is a compact self-adjoint operator on $\mathcal{H}$. Denote by $\{\lambda_i\}_{i\in\mathbb{N}}$ the spectrum of X, composed of real numbers such that $\sum_{i\in\mathbb{N}} \lambda_i^2 < +\infty$. The spectrum of D_X acting on $\mathfrak{gl}_2(\mathcal{H})$ is then the set $\{\lambda_i - \lambda_j, i, j \in \mathbb{N}\}$, and the spectrum of τ_X is the set:

$$\left\{ \frac{\sinh(\frac{\lambda_i-\lambda_j}{2})}{\frac{(\lambda_i-\lambda_j)}{2}}, i, j \in \mathbb{N} \right\}.$$

Since

$$1 \leq \frac{\sinh(\frac{\lambda_i-\lambda_j}{2})}{\frac{(\lambda_i-\lambda_j)}{2}} \leq \frac{\sinh 2\|X\|_2}{\|X\|_2},$$

it follows that τ_X is one-one on $\mathcal{S}_2(\mathcal{H})$ and bounded. Since the map that takes a formal series $f(\mathrm{x})$ to the operator $f(D_X)$ of $B(\mathcal{H})$ is a morphism of rings, the inverse of τ_X is the operator given by:

$$\begin{array}{cccc} \tau_X^{-1}: & \mathcal{S}_2(\mathcal{H}) & \rightarrow & \mathcal{S}_2(\mathcal{H}) \\ & Y & \mapsto & \frac{D_X/2}{\sinh D_X/2}(Y), \end{array}$$

whose norm is bounded by 1. Thus τ_X is an isomorphism of $\mathcal{S}_2(\mathcal{H})$ as well as $d_X \exp$. This implies that exp is a local diffeomorphism on $\mathcal{S}_2(\mathcal{H})$. Moreover, since every p in $\mathcal{P}$ admits an orthogonal basis of eigenvectors with positive eigenvalues, the exponential map from $\mathcal{S}_2(\mathcal{H})$ to $\mathcal{P}$ is one-one and onto, the inverse mapping being given by the logarithm. Therefore exp is a diffeomorphism from $\mathcal{S}_2(\mathcal{H})$ onto $\mathcal{P}$.
□

Definition 1.8 Let G be a Banach Lie group. An homogeneous space $M = G/K$ is called *reductive* if the Lie algebra $\mathfrak{g}$ of G splits into a direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{k}$ is the Lie algebra of K, and $\mathfrak{m}$ an $\mathrm{Ad}(K)$-invariant complement. A reductive homogeneous space is called *locally symmetric* if the commutation relation $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ holds.

A locally symmetric space is a particular case of a *naturally reductive* space (see Definition 7.84 page 196 in [4], Definition 23 page 312 in [17], or Proposition 5.2 page 125 in [2] and the definition that follows). In the finite-dimensional setting, the geodesics of a naturally reductive space are orbits of one-parameter subgroups of G (see Proposition 25 page 313 in [17] for a proof of this fact). The symmetric case is also treated in Theorem 3.3 page 173 in [11]. Its infinite-dimensional version has been given in Example 3.9 in [16]. The proof we give below is based on the notion of homogeneous connection.

Proposition 1.9 *Let $M = G/K$ be a locally symmetric homogeneous space under a Banach Lie group G. Then, a geodesic of M starting at $o = eK$ is given by*

$$\gamma(t) = (\exp t\mathfrak{a}) \cdot o, \qquad \mathfrak{a} \in \mathfrak{m}.$$

□ **Proof of Proposition** 1.9:
Every element $\mathfrak{a}$ in $\mathfrak{g}$ generates a vector field $X^{\mathfrak{a}}$ on the homogeneous space $M = G/K$. For every $x = g \cdot o$, $g \in G$, the Lie algebra $\mathfrak{g}$ splits into $\mathfrak{g} = \mathfrak{k}_x \oplus \mathfrak{m}_x$, where $\mathfrak{k}_x := \mathrm{Ad}(g)(\mathfrak{k})$ is the Lie algebra of the isotropy group at x and where $\mathfrak{m}_x := \mathrm{Ad}(g)(\mathfrak{m})$ can be identified with the tangent space T_xM of M at x by the application $\mathfrak{a} \mapsto X^{\mathfrak{a}}(x)$. The homogeneous connection $\hat{\nabla}$ on the tangent space of M is defined as follows. For every element $\mathfrak{a}$ in $\mathfrak{m}_x$ and every vector field X on M, one has

$$\hat{\nabla}_{X^{\mathfrak{a}}(x)} X = (\mathcal{L}_{X^{\mathfrak{a}}} X)(x) = [X^{\mathfrak{a}}, X]_{\mathfrak{X}} \tag{6}$$

where $\mathcal{L}$ denotes the Lie derivative and $[\cdot\,,\,\cdot]_{\mathfrak{X}}$ the bracket of vector fields. (The reader can check that (6) defines indeed a connection on the tangent bundle of M.) For $\mathfrak{a}$ in $\mathfrak{m}_x$ and $\mathfrak{b}$ in $\mathfrak{g}$, one has :

$$\hat{\nabla}_{X^{\mathfrak{a}}(x)} X^{\mathfrak{b}} = [X^{\mathfrak{a}}, X^{\mathfrak{b}}]_{\mathfrak{X}}(x) = -X^{[\mathfrak{a},\mathfrak{b}]}(x).$$

The torsion of the connection $\hat{\nabla}$ is given by

$$T^{\hat{\nabla}}(X^{\mathfrak{a}}, X^{\mathfrak{b}}) = \hat{\nabla}_{X^{\mathfrak{b}}} X^{\mathfrak{a}} - \hat{\nabla}_{X^{\mathfrak{a}}} X^{\mathfrak{b}} - [X^{\mathfrak{a}}, X^{\mathfrak{b}}]_{\mathfrak{X}} = -X^{[\mathfrak{a},\mathfrak{b}]}.$$

It follows that for a locally symmetric homogeneous space, the homogeneous connection is torsion free since for $\mathfrak{a}$ and $\mathfrak{b}$ in $T_xM = \mathfrak{m}_x$, $[\mathfrak{a}, \mathfrak{b}]$ belongs to the isotropy $\mathfrak{k}_x$ thus $X^{[\mathfrak{a},\mathfrak{b}]}$ vanishes. On the other hand, it follows from definition (6) that the covariant derivation of any tensor field Φ along $Y \in T_xM$ is the Lie derivative of Φ along the vector field $X^{\mathfrak{a}}$ where $\mathfrak{a} \in \mathfrak{m}_x$ is such that $Y = X^{\mathfrak{a}}(x)$. Thus the homogeneous connection preserves every G-invariant Riemannian metric. Consequently $\hat{\nabla}$ is the Levi-Civita connection of every G-invariant Riemannian metric on $M = G/H$. To see that for $\mathfrak{a} \in \mathfrak{m}$, the curve

$$\gamma(t) = (\exp t\mathfrak{a}) \cdot o, \qquad \mathfrak{a} \in \mathfrak{m}.$$

is a geodesic, note that the equality $\mathfrak{a} = \mathrm{Ad}(\exp t\mathfrak{a})(\mathfrak{a})$ implies that $\mathfrak{a}$ belongs to the space $\mathfrak{m}_{\gamma(t)}$ for all t. Hence from $\dot{\gamma}(t) = X^{\mathfrak{a}}(\gamma(t))$ it follows that $\hat{\nabla}_{\dot{\gamma}(t)}\dot{\gamma}(t) = \mathcal{L}_{X^{\mathfrak{a}}}X^{\mathfrak{a}}(\gamma(t)) = 0$. In other words γ is a geodesic of M. □

Remark 1.10 Note that for a weak Riemannian metric on a Banach manifold, the existence of the Levi-Civita connection is not guarantied in general (Koszul formula defines a element in the dual of the tangent space, which can not be represented by a vector in general). Hence the symmetric homogeneous spaces form a class of Banach manifolds for which the Levi-Civita connection exists.

Lemma 1.11 *The curve $\gamma(t) := \exp(t\log(p))$, $(0 \leq t \leq 1)$ is the unique geodesic in $\mathcal{P}$ joining the identity $o = \mathrm{id}$ to the element $p \in \mathcal{P}$. More generally, the geodesic between any two points of $\mathcal{P}$ exists and is unique.*

△ **Proof of Lemma** 1.11:
This follows from the same arguments as in [14], or by the general result stated in Proposition 1.9. Indeed the commutation relation $[\mathcal{S}_2\mathcal{H}, \mathcal{S}_2(\mathcal{H})] \subset \mathfrak{u}_2(\mathcal{H})$ implies that $\mathcal{P} = \mathrm{GL}_2(\mathcal{H})/\mathrm{U}_2(\mathcal{H})$ is locally symmetric. It follows that

$$\gamma(t) := \exp(t\log(p)) = \left(\exp\left(\frac{t}{2}\log(p)\right)\right) \cdot o, \qquad (0 \leq t \leq 1),$$

is a geodesic joining $o = \mathrm{id}$ to p. The uniqueness of this geodesic follows from the injectivity of the exponential map, since every other geodesic γ_2 joining $o = \mathrm{id}$ to p is necessarily of the form $\gamma_2(t) = \exp t\dot{\gamma}_2(0)$ by uniqueness of the geodesic starting at o with velocity $\dot{\gamma}_2(0)$. By the transitive action of $\mathrm{GL}_2(\mathcal{H})$, there exists a unique geodesic $\gamma_{p,q}$ joining two points p and q, namely

$$\gamma_{p,q}(t) := p^{\frac{1}{2}} \cdot \exp t\log\left(p^{-\frac{1}{2}} \cdot q\right) = p^{\frac{1}{2}}\left(\exp t\log\left(p^{-\frac{1}{2}}qp^{-\frac{1}{2}}\right)\right)p^{\frac{1}{2}}.$$

△

The following two Lemmas are standard results in the geometry of non-positively curved spaces.

Lemma 1.12 *The Riemannian angle between two paths f and g intersecting at o is equal to the Euclidian angle between the two paths $\log(f)$ and $\log(g)$ at 0. Moreover, in any geodesic triangle ABC in $\mathcal{P}$,*

$$c^2 \geq a^2 + b^2 - 2ab\cos\widehat{ACB},$$

where a, b, c are the lengths of the sides opposite A, B, C and $\widehat{ACB}$ the angle at A.

△ **Proof of Lemma** 1.12:
This follows from the same arguments as in [14]. The Al-Kashi inequality is also a direct consequence of Corollary 13.2 in [11] page 73, since Lemma 1.11 implies that $\mathcal{P}$ is a minimizing convex normal ball. △

Lemma 1.13 *Let $\gamma_1(t)$ and $\gamma_2(t)$ be two constant speed geodesics in $\mathcal{P}$. Then the distance in $\mathcal{P}$ between $\gamma_1(t)$ and $\gamma_2(t)$ is a convex function of t.*

△

1.3 Geodesic subspaces of $\mathcal{P}$

The following Theorem can be found in [14] in the finite-dimensional case. It work as well in the infinite-dimensional setting under an additional topological hypothesis on E, which is that E should be closed.

Theorem 1.14 ([14]) *Let E be a* closed *subspace of $\mathcal{S}_2(\mathcal{H})$. The following assertions are equivalent :*

1. $[X,[X,Y]] \in E$ *for all* $X, Y \in E$,
2. $efe \in \exp E$ *for all* $e, f \in \exp E$,
3. $\exp E$ *is a closed totally geodesic subspace of* $\mathcal{P}$.

Lemma 1.15 *:*
Let X be an element of $\mathcal{S}_2(\mathcal{H})$. Define the following maps

$$a_X(A) = A\cdot\exp X + \exp X\cdot A$$

and $\gamma_X = (d_X \exp)^{-1} \circ a_X$. *Then* $\gamma_X = D_X \coth(D_X/2)$.

$\triangle$ **Proof of Lemma** 1.15:
One has $a_X = e^{R_X} + e^{L_X}$, where R_X denotes right multiplication by X. By Proposition 1.6,

$$(d_X \exp) = e^{L_X}\left(\frac{1-e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}\right) = e^{L_X} e^{-\frac{D_X}{2}} \frac{\sinh(\frac{D_X}{2})}{D_X/2} = e^{L_X} e^{\frac{R_X - L_X}{2}} \frac{\sinh(\frac{D_X}{2})}{D_X/2} = e^{\frac{L_X}{2}} e^{\frac{R_X}{2}} \frac{\sinh(\frac{D_X}{2})}{D_X/2}.$$

Hence

$$(d_X \exp)^{-1} = e^{-\frac{L_X}{2}} e^{-\frac{R_X}{2}} \frac{D_X/2}{\sinh(\frac{D_X}{2})}.$$

It follows that

$$\gamma_X = \frac{D_X}{\sinh(\frac{D_X}{2})} e^{-\frac{L_X}{2}} e^{-\frac{R_X}{2}} \frac{e^{R_X} + e^{L_X}}{2} = \frac{D_X}{\sinh(\frac{D_X}{2})} \frac{e^{\frac{R_X - L_X}{2}} + e^{\frac{L_X - R_X}{2}}}{2} = D_X \coth(D_X/2).$$

$\triangle$

■ **Proof of Theorem** 1.14:
$1 \Rightarrow 2$: Suppose that $[X,[X,Y]] \in E$ for all X, Y in E. Let f be an element in $\exp E$, Y an element in E and X the differentiable curve in $S_2(\mathcal{H})$ defined by

$$X(t) = \log(\exp tY.f.\exp tY).$$

Let us show that $\exp X(t) = \exp tY.f.\exp tY$ belongs to $\exp E$ for every t. One has

$$\frac{d}{dt}_{|t=t_0} \exp X(t) = Y \exp X(t) + \exp X(t) Y,$$

hence $X(t)$ satisfies the following differential equation :

$$\dot{X}(t) = (d_X \exp)^{-1}_{X(t)} a_{X(t)}(Y) = \gamma_{X(t)}(Y) = D_{X(t)} \coth(D_X(t)/2)(Y).$$

Note that only even powers of D are involved in the operator $D\coth(D/2)$. Whence $X(t)$ belongs to the Banach space E as soon as $X(0) \in E$. Moreover the flow of this vector field is defined for all $t \in \mathbb{R}$. Thus setting $t = 1$ and $Y = \log e$ with $e \in \exp E$, give the result $e.f.e \in \exp E$.

$2 \Rightarrow 3$: Suppose that for all e and f in $\exp E$, the product $e.f.e$ belongs to $\exp E$. It follows from Lemma 1.11 that $\exp E$ contains every geodesic joining id to an arbitrary element in $\exp E$. Since the set of isometries of the form $x \mapsto e.x.e$ with $e \in \exp E$ fixes $\exp E$ and acts transitively on it, every geodesic joining two points of $\exp E$ is contained in $\exp E$. In other words, the space $\exp E$ is totally geodesic in $\mathcal{P}$.

$3 \Rightarrow 2$: Suppose that $\exp E$ is a closed totally geodesic subspace of $\mathcal{P}$. Let us consider the symmetry s_p with respect to $p \in \mathcal{P}$ defined from $\mathcal{P}$ to $\mathcal{P}$ by $s_p : x \mapsto px^{-1}p$ with $p \in \mathcal{P}$. Every geodesic of the form $t \mapsto p^{-\frac{1}{2}} \exp(tX) p^{-\frac{1}{2}}$ is mapped to $t \mapsto p^{-\frac{1}{2}} \exp(-tX) p^{-\frac{1}{2}}$ by s_p. It follows that every geodesic containing p is stable under s_p. Consequently if $\exp E$ is a totally geodesic subspace of $\mathcal{P}$, then

$s_p(\exp E) \subset \exp E$ for every $p \in \exp E$. Let τ_p denote the isometry of $\mathcal{P}$ defined by $\tau_p(x) = p^{\frac{1}{2}} x p^{\frac{1}{2}}$. Then

$$s_p.s_{p^{\frac{1}{2}}}(x) = p.(p^{\frac{1}{2}} x^{-1} p^{\frac{1}{2}})^{-1}.p = p^{\frac{1}{2}} x p^{\frac{1}{2}} = \tau_p(x).$$

Whence for every e, f in $\exp E$, $e.f.e = \tau_e(f) = \sigma_e(\sigma_{e^{\frac{1}{2}}}(f)) \in \exp E$.

$2 \Rightarrow 1$: Suppose that $e.f.e \in \exp E$ as soon as e, f belong to $\exp E$. For f in $\exp E$ and Y in E, let X be the differentiable curve in $\mathcal{S}_2(\mathcal{H})$ defined by

$$X(t) = \log(\exp tY.f.\exp tY).$$

Then $X(t)$ belongs to E for all $t \in \mathbb{R}$, as well as its derivative $\dot{X}(t)$. Therefore

$$Z = \lim_{t \mapsto 0} \frac{\dot{X}(t) - \dot{X}(0)}{t^2}$$

belongs to E also. Since $\dot{X}(t) = D_{X(t)} \coth(D_{X(t)}/2)(Y)$, one has

$$Z = \lim_{t \mapsto 0} \left[\frac{(1 + (1/12)t^2 D^2_{X(t)})Y - Y}{t^2} + tW \right] = (1/12)t^2 D^2_{X(0)} Y,$$

where W depends continuously on t. Hence $D_{X(0)}Y$ belongs to E. Tacking $f = \exp X$ gives the result.
■

1.4 Orthogonal projection on a geodesic subspace

In the following, E is a closed linear subspace of $\mathcal{S}_2(\mathcal{H})$ such that $[X, [X, Y]] \in E$, for all $X, Y \in E$. From corollary 1.7, it follows that $\exp E$ is closed in $\mathcal{P}$.

The proof of Mostow's decomposition theorem given in [14] is based on the existence of an orthogonal projection from $\mathcal{P}$ onto $\exp E$ which follows from compactness arguments that can't be used in the infinite dimensional setting. Here we use the completeness of $\exp E$ to obtain an analogous result.

Theorem 1.16 *There exist a continuous orthogonal projection from $\mathcal{P}$ onto $\exp E$, i.e. a continuous map π satisfying dist$(p, \exp E) =$ dist$(p, \pi(p))$ and such that the geodesic joining p to $\pi(p)$ is orthogonal to every geodesic starting from $\pi(p)$ and included in $\exp E$.*

■ **Proof of Theorem** 1.16:
Let p be a element of $\mathcal{P}$. Denote by δ the distance between p and $\exp E$ in $\mathcal{P}$ and let $\{e_n\}_{n \in \mathbb{N}}$ be a sequence in $\exp E$ thus that

$$\mathrm{dist}(p, e_n)^2 \leq \delta^2 + \frac{1}{n}.$$

Let us show that $\{e_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\exp E$. For this purpose, consider for $k > n$ the geodesic $\gamma(t)$ joining $e_n =: \gamma(0)$ to $e_k := \gamma(1)$. This geodesic lies in $\exp E$ since $\exp E$ is a geodesic subspace of $\mathcal{P}$, and is of the form:

$$\gamma(t) = e_1^{\frac{1}{2}} \exp(tH) e_1^{\frac{1}{2}},$$

where H belongs to E. Denote by $e_{n,k}$ the middle of the geodesic joining e_n to e_k, i.e. $e_{n,k} = e_1^{\frac{1}{2}} \exp(\frac{1}{2}H) e_1^{\frac{1}{2}}$. By lemma 1.12 applied to the geodesic triangle joining p, e_n and $e_{n,k}$, we have:

$$\mathrm{dist}(p, e_n)^2 \geq \mathrm{dist}(e_n, e_{n,k})^2 + \mathrm{dist}(e_{n,k}, p)^2 - 2\mathrm{dist}(e_n, e_{n,k})\mathrm{dist}(e_{n,k}, p) \cos \widehat{e_n e_{n,k} p}.$$

On the other hand, lemma 1.12 applied to the geodesic triangle joining p, e_k and $e_{n,k}$ gives:

$$\mathrm{dist}(p, e_k)^2 \geq \mathrm{dist}(e_k, e_{n,k})^2 + \mathrm{dist}(e_{n,k}, p)^2 - 2\mathrm{dist}(e_k, e_{n,k})\mathrm{dist}(e_{n,k}, p) \cos \widehat{e_k e_{n,k} p}.$$

By definition of $e_{n,k}$ we have: $\mathrm{dist}(e_k, e_{n,k}) = \mathrm{dist}(e_n, e_{n,k})$. Moreover since the geodesic γ is a smooth curve:

$$\widehat{e_k e_{n,k} p} + \widehat{e_n e_{n,k} p} = 180°,$$

and $\cos \widehat{e_k e_{n,k} p} = -\cos \widehat{e_n e_{n,k} p}$. Summing both inequalities, we obtain:

$$\mathrm{dist}(p, e_n)^2 + \mathrm{dist}(p, e_k)^2 \geq 2\mathrm{dist}(e_k, e_{n,k})^2 + 2\mathrm{dist}(e_{n,k}, p)^2.$$

It follows that:

$$\begin{aligned}\mathrm{dist}(e_k, e_{n,k})^2 &\leq \tfrac{1}{2}(\mathrm{dist}(p, e_n)^2 + \mathrm{dist}(p, e_k)^2) - \mathrm{dist}(e_{n,k}, p)^2 \\ &\leq \tfrac{1}{2}(\delta^2 + \tfrac{1}{n} + \delta^2 + \tfrac{1}{k}) - \delta^2 \\ &\leq \tfrac{1}{2}(\tfrac{1}{n} + \tfrac{1}{k}).\end{aligned}$$

This yields that $\mathrm{dist}(e_n, e_k) \leq \sqrt{2}(\frac{1}{n} + \frac{1}{k})^{\frac{1}{2}}$ and $\{e_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\exp E$. Since $\exp E$ is closed in the complete space $\mathcal{S}_2(\mathcal{H})$, the sequence $\{e_n\}_{n\in\mathbb{N}}$ converges to a element $\pi(p)$ in $\exp E$ satisfying:

$$\mathrm{dist}(p, \pi(p)) = \mathrm{dist}(p, \exp E).$$

Denote by $\alpha(t)$ the constant speed geodesic which satisfies $\alpha(0) = \pi(p)$ and $\alpha(1) = p$. By uniqueness of the geodesic joining two points it follows that the length of α is $\mathrm{dist}(p, \exp E)$. The map $x \mapsto (\pi(p))^{-\frac{1}{2}} x (\pi(p))^{-\frac{1}{2}}$ being an isometry, the curve $(\pi(p))^{-\frac{1}{2}} \alpha (\pi(p))^{-\frac{1}{2}}$ is a geodesic whose length is the distance between $(\pi(p))^{-\frac{1}{2}} p (\pi(p))^{-\frac{1}{2}}$ and $\exp E$, thus the projection of $(\pi(p))^{-\frac{1}{2}} p (\pi(p))^{-\frac{1}{2}}$ onto $\exp E$ is o. From lemma 1.11 it follows that:

$$(\pi(p))^{-\frac{1}{2}} \alpha(t) (\pi(p))^{-\frac{1}{2}} = \exp tV,$$

for some V in $\mathcal{S}_2(\mathcal{H})$. Since the length of $t \mapsto \exp tV$ is $\|V\|$, V is in F and $(\pi(p))^{-\frac{1}{2}} p (\pi(p))^{-\frac{1}{2}}$ is in $\exp F$. Since $E \perp F$, by lemma 1.12, $(\pi(p))^{-\frac{1}{2}} \alpha (\pi(p))^{-\frac{1}{2}}$ is orthogonal at the identity to every curve starting at the identity and contained in $\exp E$. Therefore α is orthogonal at $\pi(p)$ to every curve starting at $\pi(p)$ and contained in $\exp E$.

To show that π is continuous, denote by $\gamma(t)$ (resp. $\alpha(t)$) the geodesic joining a points p_1 (resp. p_2) in $\mathcal{P}$ to its projection on $\exp E$, with $\gamma(0) = \pi(p_1)$ (resp. $\alpha(0) = \pi(p_2)$) and $\gamma(1) = p_1$ (resp. $\alpha(1) = p_2$). By the negative curvature property stated in Lemma 1.13, the map $t \mapsto \mathrm{dist}(\gamma(t), \alpha(t))$ is convex. Since, for $t = 0$, $\gamma(t)$ and $\alpha(t)$ are orthogonal to the geodesic joining $\pi(p_1)$ and $\pi(p_2)$, the minimum of the distance between $\gamma(t)$ and $\alpha(t)$ is reached for $t = 0$, and $\mathrm{dist}(p_1, p_2) \geq \mathrm{dist}(\pi(p_1), \pi(p_2))$. ■

1.5 Proof of Mostow's Decomposition Theorem

Theorem 1.17 *Let E be a closed linear subspace of $\mathcal{S}_2(\mathcal{H})$ such that:*

$$[X, [X, Y]] \in E, \qquad \textit{for all} \quad X, Y \in E,$$

and let F be its orthogonal in $\mathcal{S}_2(\mathcal{H})$:

$$F := E^{\perp} = \{\ X \in \mathcal{S}_2(\mathcal{H}) \mid \mathrm{Tr}\, XY = 0, \quad \forall Y \in E\ \}.$$

For all self-adjoint positive-definite operator A in $\mathcal{S}_2(\mathcal{H})$, there exist a unique element $e \in \exp E$ and a unique element $f \in \exp F$ such that $A = efe$. Moreover the map defined from $\mathcal{P}$ to $\exp E \times \exp F$ taking A to (e, f) is a homeomorphism.

■ **Proof of Theorem** 1.17:
Denote by Υ the map from $\exp E \times \exp F$ to $\mathcal{P}$ that takes (e, f) to efe.

Let us show that Υ is one-one. Suppose that (e_1, f_1) and (e_2, f_2) are elements of $\exp E \times \exp F$ such that $e_1 f_1 e_1 = e_2 f_2 e_2$. Consider the geodesic triangle joining $e_1 f_1 e_1$, e_1^2 and e_2^2. By Theorem 1.14, $\exp E$ is a geodesic subspace of $\mathcal{P}$. Thus the geodesic joining e_1^2 to e_2^2 lies in $\exp E$. On the other hand the geodesic joining $e_1 f_1 e_1$ to e_1^2 lies in $e_1 \exp F\, e_1$. Since E is perpendicular to F at zero, $\exp E$ is perpendicular to $\exp F$ at the identity by lemma 1.12. Now the map taking x to $e_1 x\, e_1$ is an isometry, thus $e_1 \exp F\, e_1$ is perpendicular to $e_1 \exp E\, e_1 = \exp E$ at e_1^2. Hence the angle at e_1^2 of the above geodesic triangle is $90°$. Similarly, the angle at e_2^2 is $90°$ since it is formed by the geodesic joining e_2^2 to $e_2 f_2 e_2 = e_1 f_1 e_1$ which lies in $e_2 \exp F\, e_2$ and the geodesic joining e_2^2 to e_1^2 which lies in $\exp E$. Denoting by a the length of the side of the geodesic triangle joining e_1^2 to e_2^2, b the length of the side joining $e_1 f_1 e_1$ to e_1^2 and c the length of the side joining $e_1 f_1 e_1$ to e_2^2, ones has $c^2 \geq b^2 + a^2$ and $b^2 \geq c^2 + a^2$ by lemma 1.12. This implies that $a = 0$ and $e_1^2 = e_2^2$. It follows that $e_1 = e_2$ and $f_1 = f_2$.

Let us show that Υ is onto. Consider p in $\mathcal{P}$. By Theorem 1.16, the geodesic joining p to $\pi(p) \in \exp E$ is orthogonal to every geodesic starting at $\pi(p)$ and contained in $\exp E$. Denote by γ the geodesic satisfying $\gamma(0) = o$ and $\gamma(1) = (\pi(p))^{-\frac{1}{2}} p (\pi(p))^{-\frac{1}{2}}$. Since $x \mapsto (\pi(p))^{-\frac{1}{2}} x (\pi(p))^{-\frac{1}{2}}$ is an isometry, γ is orthogonal to every geodesic starting at the identity and contained in $(\pi(p))^{-\frac{1}{2}} \exp E (\pi(p))^{-\frac{1}{2}} = \exp E$. By lemma 1.12, γ is tangent to $F = E^{\perp}$ at the identity and since it is of the form $t \mapsto \exp tH$ by lemma

1.11, we have H in F. It follows that $\gamma(1) = \exp H = (\pi(p))^{-\frac{1}{2}} p (\pi(p))^{-\frac{1}{2}}$ is in $\exp F$. Therefore $p = efe$ with $e := (\pi(p))^{-\frac{1}{2}}$ in $\exp E$ and $f := (\pi(p))^{-\frac{1}{2}} p (\pi(p))^{-\frac{1}{2}}$ in $\exp F$ and Υ is onto.

The continuity of the map that takes p to $(e, f) \in \exp E \times \exp F$ with $p = efe$ follows directly from the continuity of the projection π. ■

Theorem 1.18 (Mostow's Decomposition) *Let E and F be as in Theorem 1.17. Then $GL_2(\mathcal{H})$ is homeomorphic to the product $U_2(\mathcal{H}) \cdot \exp F \cdot \exp E$.*

■ **Proof of Theorem** 1.18:
Denote by Θ the map from $U_2(\mathcal{H}) \times \exp E \times \exp F$ to $\mathrm{GL}_2(\mathcal{H})$ that takes (k, f, e) to kfe.

Let us show that Θ is one-one. Suppose that $a = k_1 f_1 e_1 = k_2 f_2 e_2$ with (k_1, f_1, e_1) and (k_2, f_2, e_2) in $U_2(\mathcal{H}) \times \exp E \times \exp F$. We have

$$a^* a = e_1 f_1^2 e_1 = e_2 f_2^2 e_2.$$

Since f_1^2 and f_2^2 are in $\exp F$, by Theorem 1.17, it follows that $e_1 = e_2$ and $f_1^2 = f_2^2$. Thus $f_1 = f_2$ and $k_1 = k_2$.

Let us show that Θ is onto. Consider x in $\mathrm{GL}_2(\mathcal{H})$. $x^* x$ is an element of $\mathcal{P}$ and by Theorem 1.17, there exist $e \in \exp E$ and $f \in \exp f$ such that $x^* x = e f^2 e$. Let k be $x(fe)^{-1}$. We have:

$$k^* k = (fe)^{-1*} x^* x (fe)^{-1} = f^{-1} e^{-1} e f^2 e e^{-1} f^{-1} = \text{ id }.$$

Thus k is in $U_2(\mathcal{H})$ and $x = kfe$.

The continuity of the map that takes x in $\mathrm{GL}_2(\mathcal{H})$ to (k, f, e) in $U_2(\mathcal{H}) \times \exp F \times \exp E$ follows from the continuity of the map that takes x to $x^* x$ and from theorem 1.17. ■

Theorem 1.19 (= Theorem 1.1) *Let G be a semi-simple connected L^*-group of compact type with Lie algebra $\mathfrak{g}$, $G^{\mathbb{C}}$ the connected L^*-group with Lie algebra $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \oplus i\mathfrak{g}$, E a closed subspace of $i\mathfrak{g}$ such that*

$$[X, [X, Y]] \in E, \qquad \text{for all} \quad X, Y \in E,$$

and F the orthogonal of E in $i\mathfrak{g}$. Then $G^{\mathbb{C}}$ is homeomorphic to the product $G \cdot \exp F \cdot \exp E$.

■ **Proof of Theorem** 1.19:
Since $\mathfrak{g}^{\mathbb{C}}$ is a semi-simple L^*-algebra, it decomposes into a Hilbert sum of closed $*$-stable simple ideals $\mathfrak{g}_j$, for j in some set J (see [18]). Since every simple L^*-algebra $\mathfrak{g}_j$ is a subalgebra of the L^*-algebra of Hilbert-Schmidt operators on some Hilbert space $\mathcal{H}_j$, $\mathfrak{g}^{\mathbb{C}}$ is a L^*-subalgebra of $\mathfrak{gl}_2(\mathcal{H})$ where $\mathcal{H}$ is the Hilbert sum of $\mathcal{H}_j$, $j \in J$. The complex L^*-group $G^{\mathbb{C}}$ is therefore an L^*-subgroup of $\mathrm{GL}_2(\mathcal{H})$. Since $G^{\mathbb{C}}$ is connected, $G^{\mathbb{C}}$ is generated by a neighborhood of the unit element. The exponential mapping is a local diffeomorphism from a neighborhood of 0 in $\mathfrak{g}^{\mathbb{C}}$ onto a neighborhood of the unit in $G^{\mathbb{C}}$. Since $\mathfrak{g}^{\mathbb{C}}$ is $*$-stable and since $(\exp X)^* = \exp X^*$, $G^{\mathbb{C}}$ is also $*$-stable. Let $x \in G^{\mathbb{C}}$. Mostow's Decomposition Theorem implies that x can be written as $x = k.f.e$ with $k \in \mathrm{U}_2(\mathcal{H})$, $e \in \exp E$ and $f \in \exp F'$, where F' is the orthogonal of E in $\mathcal{S}_2(\mathcal{H})$. Since $G^{\mathbb{C}}$ is $*$-stable, it contains $x^* x = e f^2 e$. But $x^* x$ is an Hermitian positive-definite operator in $G^{\mathbb{C}}$, thus belongs to $\exp \mathcal{S}_2(\mathcal{H}) \cap G^{\mathbb{C}} = \exp i\mathfrak{g}$. Since $f^2 = e^{-1} x^* x e^{-1}$, f^2 belongs also to $\exp i\mathfrak{g}$ hence to $\exp F$, as well as its square root f. It follows that $k = x e^{-1} f^{-1}$ belongs to $G^{\mathbb{C}} \cap \mathrm{U}_2(\mathcal{H}) = G$. ■

Corollary 1.20 *Let G be a connected semi-simple L^*-group of compact type with Lie algebra $\mathfrak{g}$ and $G^{\mathbb{C}}$ the connected L^*-group with Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$. Given an L^*-subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, denote by $\mathfrak{m}$ the orthogonal of $\mathfrak{k}$ in $\mathfrak{g}$. Then $G^{\mathbb{C}}$ is homeomorphic to $G \cdot \exp i\mathfrak{m} \cdot \exp i\mathfrak{k}$.*

□ **Proof of Corollary** 1.20:
This is a direct consequence of Theorem 1.19 since $[i\mathfrak{k}, [i\mathfrak{k}, i\mathfrak{k}]] \subset i\mathfrak{k}$. □

2 Complexification and cotangent bundle of coadjoint orbits

2.1 Finite-dimensional Theorem

Mostow's Decomposition Theorem implies the following.

Theorem 2.1 *Let G be a semi-simple compact Lie-group with Lie algebra $\mathfrak{g}$ and denote by $G^{\mathbb{C}}$ the unique complex group with Lie algebra $\mathfrak{g} := \mathfrak{g} \oplus i\mathfrak{g}$ such that G injects into $G^{\mathbb{C}}$ and such that this injection induces the natural injection $\mathfrak{g} \hookrightarrow \mathfrak{g}^{\mathbb{C}}$. Let x be an element in $\mathfrak{g}$. Denote by $\mathcal{O}_x$ the adjoint orbit of x under G, by $\mathcal{O}_x^{\mathbb{C}}$ the adjoint orbit of x under $G^{\mathbb{C}}$, and by $p\ : T\mathcal{O}_x \rightarrow \mathcal{O}_x$ the tangent bundle of the compact orbit $\mathcal{O}_x$. Then there exists a G-equivariant projection $\pi\ : \mathcal{O}_x^{\mathbb{C}} \twoheadrightarrow \mathcal{O}_x$ and a G-equivariant homeomorphism Φ from the tangent space $T\mathcal{O}_x$ onto the complex orbit $\mathcal{O}_x^{\mathbb{C}}$ which commutes with the projections p and π.*

Lemma 2.2 *Let z be an element of the compact adjoint orbit $\mathcal{O}_x$. Denote by $\mathfrak{k}_z$ the Lie algebra of the stabilizer of z, and by $\mathfrak{m}_z$ the orthogonal of $\mathfrak{k}_z$ with respect to the Killing form of $\mathfrak{g}$. Then*

$$\exp\left(i\mathfrak{m}_z\right)\cdot z\ \cap\ \mathfrak{g}\ =\ \{z\}$$

$\triangle$ **Proof of Lemma** 2.2:
Let $\mathfrak{a} \in \mathfrak{m}_z$ be such that $\exp\left(i\mathfrak{a}\right)\cdot z$ belongs to $\mathfrak{g}$. Recall that

$$\exp\left(i\mathfrak{a}\right)\cdot z = \mathrm{Ad}\left(\exp i\mathfrak{a}\right)(z) = e^{i\mathrm{ad}\mathfrak{a}}(z) = \sum_{n=1}^{+\infty}\frac{(i\ \mathrm{ad}\mathfrak{a})^n}{n!}(z) = \cosh\left(i\ \mathrm{ad}(a)\right)(z) + \sinh\left(i\ \mathrm{ad}(a)\right)(z),$$

where $\cosh\left(i\ \mathrm{ad}(a)\right)(z)$ belongs to $\mathfrak{g}$ and $\sinh\left(i\ \mathrm{ad}(a)\right)(z)$ to $i\mathfrak{g}$. Hence the condition $\exp\left(i\mathfrak{a}\right)\cdot z \in \mathfrak{g}$ reads

$$0 = \sinh(i\ \mathrm{ad}\mathfrak{a})(z).$$

Since the operator $\mathrm{ad}(\mathfrak{a})$ is skew-symmetric with respect to the Killing form, the $\mathbb{C}$-extension $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ splits into a sum of eigenspaces $\mathfrak{g}_{\lambda_j}$ of $\mathrm{ad}(\mathfrak{a})$ with eigenvalues $i\lambda_j$, where $\lambda_j \in \mathbb{R}$. Let $z = \sum_{j\in J} z_{\lambda_j}$ be the decomposition of z with respect to the direct sum $\mathfrak{g}^{\mathbb{C}} = \sum_{j\in J}\mathfrak{g}_{\lambda_j}$. One has

$$i\ \mathrm{ad}(\mathfrak{a})(z) = -\sum_{j\in J}\lambda_j\ z_{\lambda_j}$$

and

$$\sinh(i\ \mathrm{ad}\mathfrak{a})(z) = -\sum_{j\in J}\sinh(\lambda_j)\ z_{\lambda_j}.$$

It follows that $\sinh(i\ \mathrm{ad}\mathfrak{a})(z)$ vanishes if and only if $\sinh(\lambda_j)\ z_{\lambda_j}$ vanishes for all $j \in J$, or equivalently if and only if for $\lambda_j \neq 0$, $z_{\lambda_j} = 0$. Thus z belongs to the eigenspace $\mathfrak{g}_0$ which is the kernel of $\mathrm{ad}(\mathfrak{a})$. But the equation $[\mathfrak{a}, z] = 0$ implies that $\mathfrak{a} = 0$ since $\mathfrak{a}$ belongs to $\mathfrak{m}_z$ by hypothesis. Consequently $\exp\left(i\mathfrak{m}_z\right)\cdot z\ \cap\ \mathfrak{g}$ reduces to z and the Lemma is proved. $\triangle$

■ **Proof of Theorem** 2.1:
Let us first show that every y in the complex adjoint orbit $\mathcal{O}_x^{\mathbb{C}}$ can be written uniquely as

$$y = \exp(i\mathfrak{a})\cdot z, \tag{7}$$

where z belongs to $\mathcal{O}_x$ and $\mathfrak{a}$ to $\mathfrak{m}_z$. Since y belongs to the complex orbit of x, there exists $g \in G^{\mathbb{C}}$ such that $y = g\cdot x = \mathrm{Ad}(g)(x)$. By Mostow's Decomposition Theorem, there exist $u \in G$, $\mathfrak{b} \in \mathfrak{m}_x$ and $\mathfrak{c} \in \mathfrak{k}_x$ such that $g = u\exp i\mathfrak{b}\exp i\mathfrak{c}$. It follows that $y = (u\exp i\mathfrak{b})\cdot x$ since $\exp i\mathfrak{k}_x$ acts trivially on x. But $u\exp i\mathfrak{b} = \exp\left(i\mathrm{Ad}(u)(\mathfrak{b})\right)u$. Hence $y = \exp(i\mathfrak{a})\cdot z$ with $z := u\cdot x \in \mathcal{O}_x$ and $\mathfrak{a} := \mathrm{Ad}(u)(\mathfrak{b}) \in \mathfrak{m}_z$. This proves the existence of the expression (7). In order to proof uniqueness, let us suppose that

$$\exp\left(i\mathfrak{a}_1\right)\cdot z_1 = \exp\left(i\mathfrak{a}_2\right)\cdot z_2,$$

for some z_1, z_2 in $\mathcal{O}_x$, some $\mathfrak{a}_1$ in $\mathfrak{m}_{z_1}$ and some $\mathfrak{a}_2$ in $\mathfrak{m}_{z_2}$. One has :

$$\exp\left(-i\mathfrak{a}_2\right)\exp\left(i\mathfrak{a}_1\right)\cdot z_1 = z_2. \tag{8}$$

By Mostow's Decomposition Theorem, there exists u' in G, $\mathfrak{a}_3$ in $\mathfrak{m}_{z_1}$ and $\mathfrak{b}_3$ in $\mathfrak{k}_{z_1}$ such that

$$\exp(-\mathrm{i}\mathfrak{a}_2)\exp(\mathrm{i}\mathfrak{a}_1) = u'\exp(\mathrm{i}\mathfrak{a}_3)\exp(\mathrm{i}\mathfrak{b}_3). \tag{9}$$

Thus equation (8) becomes

$$u'\exp(\mathrm{i}\mathfrak{a}_3)\cdot z_1 = z_2,$$

or equivalently

$$\exp(\mathrm{i}\mathfrak{a}_3)\cdot z_1 = (u')^{-1}\cdot z_2,$$

But $(u')^{-1}\cdot z_2$ belongs to $\mathcal{O}_x$ since z_2 is an element of $\mathcal{O}_x$ and $u'\in G$. Whence Lemma 2.2 implies that

$$\exp(\mathrm{i}\mathfrak{a}_3)\cdot z_1 = (u')^{-1}\cdot z_2 = z_1.$$

It follows that $\mathfrak{a}_3$ stabilizes z_1, hence vanishes because it belongs to $\mathfrak{k}_{z_1}\cap\mathfrak{m}_{z_1}=\{0\}$, and that $z_2 = u'\cdot z_1$. Therefore equation (9) becomes

$$\exp(-\mathrm{i}\mathfrak{a}_2)\exp(\mathrm{i}\mathfrak{a}_1) = u'\exp(\mathrm{i}\mathfrak{b}_3).$$

It follows that

$$\exp(\mathrm{i}\mathfrak{a}_1) = \exp(\mathrm{i}\mathfrak{a}_2)\,u'\exp(\mathrm{i}\mathfrak{b}_3) = u'\;\exp\left(\mathrm{i}\mathrm{Ad}\left((u')^{-1}\right)(\mathfrak{a}_2)\right)\;\exp(\mathrm{i}\mathfrak{b}_3),$$

where $\mathrm{Ad}\left((u')^{-1}\right)(\mathfrak{a}_2)$ belongs to $\mathrm{Ad}\left((u')^{-1}\right)(\mathfrak{m}_{z_2}) = \mathfrak{m}_{z_1}$. By uniqueness of Mostow's decomposition, one has

$$u' = e, \qquad \mathfrak{a}_2 = \mathfrak{a}_1, \qquad \text{and}\;\; \mathfrak{b}_3 = 0.$$

Thus uniqueness of (7) is proved. The projection π is defined as follows :

$$\begin{array}{rccc} \pi: & \mathcal{O}_x^{\mathbb{C}} & \twoheadrightarrow & \mathcal{O}_x \\ & y = \exp(\mathrm{i}\mathfrak{a})\cdot z & \mapsto & z, \end{array}$$

where in the expression of y, the element z is supposed to belong to $\mathcal{O}_x$ and $\mathfrak{a}$ to $\mathfrak{m}_z$. The G-equivariance of the projection π is a direct consequence of the identity

$$u\cdot y = \exp(\mathrm{i}\mathrm{Ad}(u)(\mathfrak{a}))\cdot(u\cdot z),$$

and of the relation $\mathrm{Ad}(u)(\mathfrak{m}_z) = \mathfrak{m}_{u\cdot z}$. Let us recall that the tangent space to $\mathcal{O}_x$ at z can be identified with $\mathfrak{m}_z$. Define Φ by

$$\begin{array}{rccc} \Phi: & T\mathcal{O}_x & \to & \mathcal{O}_x^{\mathbb{C}} \\ & \mathfrak{a}\in\mathfrak{m}_z = T_z\mathcal{O}_x & \mapsto & \exp(\mathrm{i}\mathfrak{a})\cdot z. \end{array}$$

The G-equivariance of Φ is clear. It is an homeomorphism of fiber bundles since Mostow's Decomposition Theorem is an homeomorphism. ■

2.2 Infinite-dimensional Theorem

An infinite-dimensional analogue of Theorem 2.1 is given in Theorem 2.3. In order to state it, let us introduce some notation. In the following, G will denote a *simple* L^*-group of compact type, that is an Hilbert Lie group such that

$$\langle [x\,,\,y]\,,\,z\rangle = -\langle y\,,\,[x\,,\,z]\rangle. \tag{10}$$

It follows from the classification Theorem of simple L^*-groups of compact type given in [3], [10], or [19], that G is a group of Hilbert-Schmidt operators on a certain Hilbert space $\mathcal{H}$. Denote by $\mathfrak{g}$ the Lie algebra of G and by $G^{\mathbb{C}}$ the complex L^*-group with Lie algebra $\mathfrak{g} := \mathfrak{g}\oplus i\mathfrak{g}$ characterized by the property that G injects into $G^{\mathbb{C}}$ and that this injection induces the natural injection $\mathfrak{g}\hookrightarrow\mathfrak{g}^{\mathbb{C}}$. According to Theorem 4.4 in [15], every derivation of $\mathfrak{g}$ *with closed range* is diagonalizable on $\mathfrak{g}^{\mathbb{C}}$ and represented by the bracket with some skew-Hermitian operator D with finite spectrum. Denote by $\mathcal{O}^{\mathbb{C}}$ the orbit of 0 under the affine adjoint action of $G^{\mathbb{C}}$ defined by

$$\begin{array}{rccc} \mathrm{Ad}_D: & G^{\mathbb{C}}\times\mathfrak{g}^{\mathbb{C}} & \longrightarrow & \mathfrak{g}^{\mathbb{C}} \\ & (g\,,\,y) & \longmapsto & g\,y\,g^{-1} + g\,D\,g^{-1} - D, \end{array}$$

and by $\mathcal{O}$ the affine adjoint orbit of 0 under the restriction of Ad_D to G.

Theorem 2.3 *There exists a G-equivariant projection π from $\mathcal{O}^{\mathbb{C}}$ onto $\mathcal{O}$ and a G-equivariant homeomorphism Φ from the tangent space to the orbit $\mathcal{O}$ of compact type onto the complex orbit $\mathcal{O}_x^{\mathbb{C}}$ which commutes with the projection p and π.*

■ **Proof of Theorem** 2.3:
Every y in $\mathcal{O}^{\mathbb{C}}$ is of the form

$$y = g\ D\ g^{-1} - D = g \cdot 0$$

for some $g \in G^{\mathbb{C}}$. Let us denote by $\mathfrak{k}$ the commutator of D in $\mathfrak{g}$ and by $\mathfrak{m}$ its orthogonal. For every $z = u \cdot 0$ in $\mathcal{O}$ (where $u \in G$), set $\mathfrak{k}_z = \mathrm{Ad}(u)(\mathfrak{k})$ and $\mathfrak{m}_z = \mathrm{Ad}(u)(\mathfrak{m})$. By Mostow's Decomposition Theorem, $G^{\mathbb{C}}$ decomposes into the product $G \cdot \exp(\mathrm{i}\ \mathfrak{m}) \cdot \exp(\mathrm{i}\ \mathfrak{k})$. Hence y can be written as

$$y = u \exp(\mathrm{i}\ \mathfrak{b}) \cdot 0 \qquad \mathfrak{b} \in \mathfrak{m}, \quad u \in G$$

or more conveniently as

$$y = \exp(\mathrm{i}\ \mathfrak{a}) \cdot z,$$

where $z = u \cdot 0$ belongs to $\mathcal{O}$ and $\mathfrak{a} = \mathrm{Ad}(u)(\mathfrak{b})$ is an element in $\mathfrak{m}_z$. To show that this expression of z is unique and defines a projection π from $\mathcal{O}^{\mathbb{C}}$ onto $\mathcal{O}$ by $\pi(y) = z$, it is sufficient to show that

$$\exp(\mathrm{i}\ \mathfrak{a}) \cdot z \cap \mathfrak{g} = \{z\} \tag{11}$$

where $\mathfrak{a}$ belongs to $\mathfrak{m}_z$(see Lemma 2.2 and Theorem 2.1). By G-equivariance, it is sufficient to show that the previous equality for $z = 0$. For $\mathfrak{a} \in \mathfrak{m}$, one has

$$y = \exp(\mathrm{i}\ \mathfrak{a})\,(D)\exp(-\mathrm{i}\ \mathfrak{a}) - D = e^{\mathrm{i}\ \mathrm{ad}(\mathfrak{a})}\,(D) - D = \cosh(\mathrm{i}\ \mathrm{ad}(\mathfrak{a}))\,(D) - D + \sinh(\mathrm{i}\ \mathrm{ad}(\mathfrak{a}))\,(D)\,,$$

where $\cosh(\mathrm{i}\ \mathrm{ad}(\mathfrak{a}))\,(D) - D$ belongs to $\mathfrak{g}$ and $\sinh(\mathrm{i}\ \mathrm{ad}(\mathfrak{a}))\,(D)$ to $\mathrm{i}\mathfrak{g}$. Hence the condition $\exp(\mathrm{i}\ \mathfrak{a}) \cdot 0 \in \mathfrak{g}$ reads

$$0 = \sinh(\mathrm{i}\ \mathrm{ad}(\mathfrak{a}))(D) = \mathrm{i}\frac{\sinh(\mathrm{i}\ \mathrm{ad}(\mathfrak{a}))}{\mathrm{i}\ \mathrm{ad}(\mathfrak{a})}([\mathfrak{a}, D]),$$

where $[\mathfrak{a}, D]$ belongs to $\mathfrak{g}$ since D is a derivation of $\mathfrak{g}$. Since $\frac{\sinh(t)}{t} \geq 1$, the condition $\sinh(\mathrm{i}\ \mathrm{ad}(\mathfrak{a}))(D) = 0$ implies that $[\mathfrak{a}, D]$ vanishes. Thus $\mathfrak{a}$ belongs to $\mathfrak{k}$. But $\mathfrak{a} \in \mathfrak{m}$ by hypothesis, so $\mathfrak{a} = 0$. Consequently equality (11) is satisfied and the projection π well-defined. The definition of the homeomorphism Φ is as in the finite-dimensional case. ■

3 Stable manifolds

In this section G will denotes a semi-simple L^*-group of compact type with Lie algebra $\mathfrak{g}$ and $G^{\mathbb{C}}$ the complex L^*-group with Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus \mathrm{i}\mathfrak{g}$ characterized by the property that the natural injection $\mathfrak{g} \hookrightarrow \mathfrak{g}^{\mathbb{C}}$ is the differential map of an injection $G \hookrightarrow G^{\mathbb{C}}$. Recall that any compact Lie group is an L^*-group, its Lie algebra being endowed with the scalar product given by the Killing form. Let us suppose that G acts on a Banach weak Kähler manifold $(M, \omega, \mathrm{g}, I)$ and preserves the Kähler structure of M. We use the following convention. The Riemannian metric g, the symplectic form ω and the complex structure I of M are related by the following formula : $\omega(\cdot\,,\,\cdot) = \mathrm{g}(I\cdot\,,\,\cdot)$. Suppose that the G-action extends to an holomorphic action of $G^{\mathbb{C}}$ on M. Let $\mu\ : M \rightarrow \mathfrak{g}'$ be a G-equivariant moment map of the G-action, where $\mathfrak{g}'$ denotes the continuous dual of $\mathfrak{g}$. By definition :

$$d\mu_x(\mathfrak{a}) = i_{X^{\mathfrak{a}}}\omega,$$

for $x \in M$, $\mathfrak{a} \in \mathfrak{g}$, where $X^{\mathfrak{a}}$ denotes the Killing vector field generated by the element $\mathfrak{a}$ in $\mathfrak{g}$. The following Lemma is an extension of Lemma 7.2 page 96 in [8].

Lemma 3.1 *Let $\xi \in \mathfrak{g}'$ be a G-invariant regular value of the moment map μ, and x be an element of the level set $\mu^{-1}(\xi)$. Denote by $\mathfrak{k}_x$ the Lie algebra of the isotropy group of x and $\mathfrak{m}_x$ the orthogonal of $\mathfrak{k}_x$ in $\mathfrak{g}$. Then*

$$\exp(\mathrm{i}\mathfrak{m}_x) \cdot x\ \cap\ \mu^{-1}(\xi) = \{x\}.$$

△ **Proof of Lemma** 3.1:
Let $\mathfrak{a}$ be an element in $\mathfrak{m}_x$ such that $\exp(\mathrm{i}\,\mathfrak{a}) \cdot x$ belongs to the level set $\mu^{-1}(\xi)$. Consider the real function f defined by

$$f(t) := \mu\,(\exp(\mathrm{i}t\mathfrak{a}) \cdot x)\,(\mathfrak{a}).$$

One has $f(0)=\mu(x)(\mathfrak{a})=\xi(\mathfrak{a})$ since x belongs to the level set $\mu^{-1}(\xi)$, and $f(1)=\mu\left(\exp(\mathrm{i}t\mathfrak{a})\cdot x\right)(\mathfrak{a})=\xi(\mathfrak{a})$ by the hypothesis on $\mathfrak{a}$. The differentiability of f implies that there exists t_0 in $(0\,,\,1)$ such that the derivative of f at t_0 vanishes. One has

$$0=h'(t_0)=(d\mu)_y\left(\mathrm{i}\mathfrak{a}\cdot y\right)(\mathfrak{a})=i_{\mathfrak{a}\cdot y}\omega(\mathrm{i}\mathfrak{a}\cdot y),$$

where we have set $y:=\exp(\mathrm{i}t_0\mathfrak{a})\cdot x$. Since the action of $G^{\mathbb{C}}$ is holomorphic, we have

$$0=h'(t_0)=i_{\mathfrak{a}\cdot y}\omega\left(I(\mathfrak{a}\cdot y)\right)=\omega\left(\mathfrak{a}\cdot y,I(\mathfrak{a}\cdot y)\right)=-\mathrm{g}(\mathfrak{a}\cdot y,\mathfrak{a}\cdot y).$$

It follows that $\mathfrak{a}\cdot y=0$. But

$$\mathfrak{a}\cdot y=\mathfrak{a}\cdot\left(\exp(\mathrm{i}t_0\mathfrak{a})\cdot x\right)=\left(\exp(\mathrm{i}t_0\mathfrak{a})\right)_*(\mathfrak{a}\cdot x),$$

hence $\mathfrak{a}\cdot x$ vanishes also and $\mathfrak{a}$ belongs to $\mathfrak{k}_x$. But by hypothesis $\mathfrak{a}$ belongs to $\mathfrak{m}_x$, thus $\mathfrak{a}$ vanishes. It follows that the intersection of $\exp(\mathrm{i}\mathfrak{m}_x)\cdot x$ with the level set is reduced to $\{x\}$. △

Definition 3.2 *The stable manifold M^s associated to the level set $\mu^{-1}(\xi)$ is defined as*

$$M^s:=\{\ y\in M\mid\exists\, g\in G^{\mathbb{C}},\ \ g\cdot y\in\mu^{-1}(\xi)\ \}.$$

Theorem 3.3 *There exists a G-equivariant projection from the stable manifold associated to $\mu^{-1}(\xi)$ onto the level set $\mu^{-1}(\xi)$.*

■ **Proof of Theorem** 3.3:
Let us prove that every element in M^s can be written uniquely as

$$y=\exp(\mathrm{i}\mathfrak{a})\cdot z,\tag{12}$$

where z belongs to the level set $\mu^{-1}(\xi)$ and $\mathfrak{a}$ to the orthogonal $\mathfrak{m}_z$ in $\mathfrak{g}$ of the isotropy algebra $\mathfrak{k}_z$ of z. By definition of the stable manifold, there exists g in $G^{\mathbb{C}}$ and x in $\mu^{-1}(\xi)$ such that $y=g\cdot x$. By Mostow's Decomposition Theorem $G^{\mathbb{C}}=G\cdot\exp(\mathrm{i}\mathfrak{m}_x)\cdot\exp(\mathrm{i}\mathfrak{k}_x)$, hence there exists $u\in G$, $\mathfrak{b}\in\mathfrak{m}_x$ and $\mathfrak{c}\in\mathfrak{k}_x$ such that $g=u\exp\mathrm{i}\mathfrak{b}\exp\mathrm{i}\mathfrak{c}$. It follows that $y=(u\exp\mathrm{i}\mathfrak{b})\cdot x$ since $\exp\mathrm{i}\mathfrak{k}_x$ acts trivially on x. But $u\exp\mathrm{i}\mathfrak{b}=\exp\left(\mathrm{i}\mathrm{Ad}(u)(\mathfrak{b})\right)u$. Hence $y=\exp(\mathrm{i}\mathfrak{a})\cdot z$ where $z:=u\cdot x$ belongs to the level set $\mu^{-1}(\xi)$ and where $\mathfrak{a}:=\mathrm{Ad}(u)(\mathfrak{b})\in\mathfrak{m}_z$ since the scalar product is G-invariant. This proves the existence of the expression (12). In order to proof uniqueness, let us suppose that

$$\exp\left(\mathrm{i}\mathfrak{a}_1\right)\cdot z_1=\exp\left(\mathrm{i}\mathfrak{a}_2\right)\cdot z_2,$$

for some z_1, z_2 in $\mu^{-1}(\xi)$, some $\mathfrak{a}_1$ in $\mathfrak{m}_{z_1}$ and some $\mathfrak{a}_2$ in $\mathfrak{m}_{z_2}$. One has :

$$\exp\left(-\mathrm{i}\mathfrak{a}_2\right)\exp\left(\mathrm{i}\mathfrak{a}_1\right)\cdot z_1=z_2.\tag{13}$$

By Mostow's Decomposition Theorem, there exists u' in G, $\mathfrak{a}_3$ in m_{z_1} and $\mathfrak{b}_3$ in $\mathfrak{k}_{z_1}$ such that

$$\exp\left(-\mathrm{i}\mathfrak{a}_2\right)\exp\left(\mathrm{i}\mathfrak{a}_1\right)=u'\exp(\mathrm{i}\mathfrak{a}_3)\exp(\mathrm{i}\mathfrak{b}_3).\tag{14}$$

Thus equation (13) becomes

$$u'\exp(\mathrm{i}\mathfrak{a}_3)\cdot z_1=z_2,$$

or equivalently

$$\exp(\mathrm{i}\mathfrak{a}_3)\cdot z_1=(u')^{-1}\cdot z_2,$$

But $(u')^{-1}\cdot z_2$ belongs to $\mu^{-1}(\xi)$ since z_2 is an element of $\mu^{-1}(\xi)$ and $u'\in G$. Whence Lemma 3.1 implies that

$$\exp(\mathrm{i}\mathfrak{a}_3)\cdot z_1=(u')^{-1}\cdot z_2=z_1.$$

It follows that $\mathfrak{a}_3$ stabilizes z_1, hence vanishes because it belongs to $\mathfrak{k}_{z_1}\cap\mathfrak{m}_{z_1}=\{0\}$, and that $z_2=u'\cdot z_1$. Therefore equation (14) becomes

$$\exp\left(-\mathrm{i}\mathfrak{a}_2\right)\exp\left(\mathrm{i}\mathfrak{a}_1\right)=u'\exp(\mathrm{i}\mathfrak{b}_3).$$

It follows that

$$\exp\left(\mathrm{i}\mathfrak{a}_1\right)=\exp\left(\mathrm{i}\mathfrak{a}_2\right)u'\exp(\mathrm{i}\mathfrak{b}_3)=u'\ \exp\left(\mathrm{i}\mathrm{Ad}\left((u')^{-1}\right)(\mathfrak{a}_2)\right)\ \exp(\mathrm{i}\mathfrak{b}_3),$$

where $\mathrm{Ad}\left((u')^{-1}\right)(\mathfrak{a}_2)$ belongs to $\mathrm{Ad}\left((u')^{-1}\right)(\mathfrak{m}_{z_2}) = \mathfrak{m}_{z_1}$. By uniqueness of Mostow's decomposition, one has

$$u' = e, \qquad \mathfrak{a}_2 = \mathfrak{a}_1, \qquad \text{and} \;\; \mathfrak{b}_3 = 0.$$

Thus uniqueness of (12) is proved. The projection π is defined as follows :

$$\begin{array}{rccl} \pi \;: & M^s & \twoheadrightarrow & \mu^{-1}(\xi) \\ & y = \exp(\mathrm{i}\mathfrak{a}) \cdot z & \mapsto & z, \end{array}$$

where in the expression of y, the element z is supposed to belong to the level set $\mu^{-1}(\xi)$ and $\mathfrak{a}$ to $\mathfrak{m}_z$. The G-equivariance of the projection π is a direct consequence of the identity

$$u \cdot y = \exp\left(\mathrm{i}\mathrm{Ad}(u)(\mathfrak{a})\right) \cdot (u \cdot z),$$

and of the relation $\mathrm{Ad}(u)(\mathfrak{m}_z) = \mathfrak{m}_{u \cdot z}$. ■

Remark 3.4 In the case where $\mathcal{O}_x$ is a (irreducible) Hermitian-symmetric coadjoint orbit of compact type, Theorem 2.1 is a particular case of Theorem 3.3 in the following sense. The coadjoint action of $G^{\mathbb{C}}$ on $\mathcal{O}_x^{\mathbb{C}}$ is holomorphic with respect to the natural complex structure of $\mathcal{O}_x^{\mathbb{C}}$. The moment map μ_1 for the G-action on $\mathcal{O}_x^{\mathbb{C}}$ corresponding to the real symplectic form ω_1 associated to the natural complex structure of $\mathcal{O}_x^{\mathbb{C}}$ has been computed in [6], p 153, formula (3.14) :

$$\mu_1(y) = -i\left[\frac{\pi(y)}{\kappa}, y\right],$$

where κ is a positive constant. It follows that the orbit of compact type $\mathcal{O}_x$ can be thought as the level set ${\mu_1}^{-1}(0)$ and $\mathcal{O}_x^{\mathbb{C}}$ as the associated stable manifold. We believe that this picture is also faithful in the case of a general orbit of compact type.

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